Mathematical Induction Assignments

Prove the Following using Principle of Mathematical induction

1) Prove that for any positive integer number n , $n^3 + 2n$ is divisible by 3
2) Prove that

   $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2 (n + 1)^2 / 4$

   for all positive integers n.

3) For every $n \in N$, $2n^3 + 3n^2 + n$ is divisible by 6.
4) Prove by induction that

   $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots n \cdot (n + 1) = n(n + 1)(n + 2)/ 3$

5) For every $n \geq 2$, n^3-n is multiple of 6
6) For every $n \geq 7$, $3^n \geq n!$
7) For all $n \geq 1$

   $(1+x)^n \geq 1+nx$

   Where $(1+x) > 0$

8) If $n \in N$, then $1\cdot3+2\cdot4+3\cdot5+4\cdot6+\cdots + n(n+2) = n(n+1)(2n+7) /6$
9) Prove that $3 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = (3^{n+1} - 3)/ 2$ for every $n \in N$.
10) Prove that $1/ 1 + 1/ 4 + 1/ 9 + \cdots + 1/ n^2 \leq 2 - 1/ n$
11) For all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Solution to Problem 1:

Let Statement P (n) is defined by

$n^3 + 2n$ is divisible by 3

Step 1: **Basic Step**

We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3

hence p (1) is true.

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

$k^3 + 2k$ is divisible by 3

is equivalent to

$k^3 + 2k = 3B$, where B is a positive integer.

Step 3: **Inductive Steps**

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3B + 3[k^2 + k + 1] = 3[B + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Solution to Problem 2:

Statement P (n) is defined by

$1^3 + 2^3 + 3^3 + \dots + n^3 = n^2 (n + 1)^2 / 4$

Step 1: **Basic Step**

We first show that p (1) is true.

Left Side $= 1^3 = 1$

Right Side = $1^2 (1 + 1)^2 / 4 = 1$

hence p (1) is true.

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

$1^3 + 2^3 + 3^3 + ... + k^3 = k^2 (k + 1)^2 / 4$

Step 3: **Inductive Steps**

add $(k + 1)^3$ to both sides

$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = k^2 (k + 1)^2 / 4 + (k + 1)^3$

factor $(k + 1)^2$ on the right side

$= (k + 1)^2 [k^2 / 4 + (k + 1)]$

set to common denominator and group

$= (k + 1)^2 [k^2 + 4 k + 4] / 4$

$= (k + 1)^2 [(k + 2)^2] / 4$

We have started from the statement P(k) and have shown that

$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = (k + 1)^2 [(k + 2)^2] / 4$

Which is the statement P(k + 1).

Solution to Problem 3:

Let P(n)

$2n^3 + 3n^2 + n$ is divisible by 6

Step 1: **Basic Step**

P(1) is just that 2 + 3 + 1 is divisible by 6, which is trivial.

STEP 2: **Inductive Hypothesis**

We now assume that P (k) is true

Ie.

$2k^3 + 3k^2 + k$ is divisible by 6

Step 3: **Inductive Steps**

We have to prove P(k+1)
Now

$2(k + 1)^3 + 3(k + 1)^2 + (k + 1)$

$= 2(k^3 + 3k^2 + 3k + 1) + 3(k^2 + 2k + 1) + (k + 1)$

$= (2k^3 + 3k^2 + k) + (6k^2 + 6k + 2 + 6k + 3 + 1)$

$= (2k^3 + 3k^2 + k) + 6(k^2 + 2k + 1)$

The first term is divisible by 6 since P(k) is true and the second term is a multiple of 6. Hence, the last quantity is divisible by 6

Solution to Problem 4:

Statement P (n) is defined by

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots n \cdot (n + 1) = n(n + 1)(n + 2)/ 3$

Step 1: **Basic Step**

We first show that p (1) is true.

Left Side = 1.2 = 2

Right Side = 1 (1 + 1)(1+2) / 3 = 2

hence p (1) is true.

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots k \cdot (k + 1) = k(k + 1)(k + 2)/ 3$

Step 3: **Inductive Steps**

We have to prove P(k+1)
Now

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots (k +1)\cdot [(k+1) + 1) = (k+1)[(k +1)+ 1][(k+1) + 2]/ 3$

Taking LHS

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots (k +1)\cdot [(k+1) + 1)$

$=1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots k(k+1) + (k +1)\cdot [(k+1) + 1)$

$= k(k + 1)(k + 2)/ 3 + (k +1)\cdot [(k+1) + 1)$

$= k(k + 1)(k + 2)/ 3 + (k +1)\cdot (k+2)$

$=(k+1)(k+2) [k/3 +1]$

$=(k+1)(k+2)(k+3)/3$

Which is the statement P(k + 1).

Solution to Problem 6:

Let Statement P (n) is defined by

for all $n \geq 7$, $n! > 3^n$

Step 1: **Basic Step**

Let $n = 7$

$n! > 3^n$
$7! = 5040$

$3^7 = 2187$

So p(7) is true

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

That is, $k! > 3^k$

Step 3: **Inductive Steps**

Let $n = k + 1$.

Then:

$(k+1)! = (k+1)k!$

$> (k+1)\, 3^k$

Now $k \geq 7$

So $(k+1) > 3$

$> 3 .\, 3^k$

$>3^{k+1}$

Then P(n) holds for $n = k + 1$, and thus for all $n \geq 7$

Solution to Problem 7:

Let Statement P (n) is defined by

$(1+x)^n \geq 1+nx$

Where $(1+x) > 0$

Step 1: **Basic Step**

Let $n = 1$

$(1+x)^n \geq 1+nx$

$(1+x) \geq 1+x$

Which is true

So p(1) is true

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

$(1+x)^k \geq 1+kx$

Step 3: **Inductive Steps**

Let $n = k + 1$.

Then:

$(1+x)^{k+1} \geq 1+(k+1)x$

Taking the LHS

$(1+x)^{k+1} = (1+x)(1+x)^k$

Now from hypothesis we know that

$(1+x)^k \geq 1+kx$

Also $(1+x) > 0$

So $(1+x)^{k+1} \geq (1+x)(1+kx)$

$\geq [1+kx^2+(k+1)x]$

Now kx^2 is a positive quantity so we can say that

$\geq [1+(k+1)x]$

Which is P(k+1)

Solution to Problem 11:

Let Statement P (n) is defined by

for all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Step 1: **Basic Step**

Let $n = 1$.

Then the expression $8^n - 3^n$ evaluates to $8^1 - 3^1 = 8 - 3 = 5$, which is clearly divisible by 5.

STEP 2: **Inductive Hypothesis**

We now assume that p (k) is true

That is, that $8^k - 3^k$ is divisible by 5.

Step 3: **Inductive Steps**

Let $n = k + 1$.

Then:

$$8^{k+1} - 3^{k+1} = 8^{k+1} - 3\times8^{k} + 3\times8^{k} - 3^{k+1}$$

$$= 8^{k}(8 - 3) + 3(8^{k} - 3^{k}) = 8^{k}(5) + 3(8^{k} - 3^{k})$$

The first term in $8^{k}(5) + 3(8^{k} - 3^{k})$ has 5 as a factor (explicitly), and the second term is divisible by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, $8^{k}(5) + 3(8^{k} - 3^{k}) = 8^{k+1} - 3^{k+1}$, must be divisible by 5.

Then P(n) holds for $n = k + 1$, and thus for all $n \geq 1$.